INPUT-OUTPUT PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

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The majority of the results describing the input-output properties of feedback systems are based on some properties of linear time-invariant systems. This is the case for derivations of the circle criterion (say, using the small gain theorem), the Popov criterion (say, using the passivity theorem), and in the use of the loop shifting theorem which is so useful to shift sectors of the form $[0, k]$ to $[k_1, k_2]$. Thus it is important to obtain most general results concerning linear time-invariant systems. We propose to present some recent developments in this field which improve upon the results of [1-4]. Furthermore we particularly emphasize the parallelism that exists between the continuous-time case and the discrete-time case.

We consider an n-input, n-output, linear time-invariant feedback system. To start with, we assume unity feedback. The open loop gain is specified by the $n \times n$ matrix transfer function $\hat{G}(s)$ in the continuous case and $\tilde{G}(z)$ in the discrete-time case.

Notations.

In the following, $\mathbb{R}$ ($\mathbb{C}$) denotes the field of real (complex) numbers. $\mathbb{R}_+$ denotes the nonnegative real numbers. $\mathbb{R}^n$ ($\mathbb{R}^{nxn}$) denotes the set of all n-vectors (nxn matrices) with elements in $\mathbb{R}$. $\mathbb{C}^n$ and $\mathbb{C}^{nxn}$ are similarly defined. For any $\sigma \in \mathbb{R}$, $\mathcal{A}(\sigma)$ denotes the Banach algebra, [1], (where "+" is the pointwise addition and product is the convolution) of generalized functions of the form:

$$f(t) = \begin{cases} f_a(t) + \sum_{i=0}^{\infty} f_i \, \delta(t-t_i) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where $t \mapsto f_a(t)e^{-\sigma t}$ is in L^1; with $0 = t_0 < t_1 < \cdots$, $f_i \in \mathbb{R}$, $\forall_i$, and $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$. $\mathcal{A}^n(\sigma)$ $(\mathcal{A}^{nxn}(\sigma))$ denotes the set of all n-vectors (nxn matrices) with components in $\mathcal{A}(\sigma)$. If $\sigma = 0$, we write $\mathcal{A}$ instead of $\mathcal{A}(0)$.

The superscript $\hat{}\ (\tilde{})$ denotes Laplace transforms: $\hat{f} = \mathcal{L}[f]$. (z-transforms: $\tilde{f} = \mathcal{Z}[f]$). For a treatment of analytic functions taking values in $\mathbb{C}^{nxn}$ see [7].

Typical among the new results that we prove are

Theorem 1. (Continuous-time) Suppose that

$$\hat{G}(s) = \hat{G}_a(s) + \sum_{i=0}^{\infty} G_i e^{-st_i} + \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m_\alpha} \frac{R_{\alpha\beta}}{(s-p_\alpha)^\beta}$$

$$\triangleq \hat{G}_\ell(s) + \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m_\alpha} \frac{R_{\alpha\beta}}{(s-p_\alpha)^\beta}$$

where

(a) $\hat{G}_\ell(\cdot) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ for some $\sigma \in \mathbb{R}$;

(b) $R_{\alpha\beta} \in \mathbb{C}^{nxn}$ for $\beta = 1, 2, \cdots, m_\alpha$, $\alpha = 1, 2, \cdots, k$

(c) for $\alpha = 1, 2, \cdots, k$, $\mathrm{Re}[p_\alpha] \geq \sigma$; and $p_\alpha \neq p_{\alpha'}$ for $\alpha \neq \alpha'$.

Under these conditions, if

(i) $\det R_{\alpha m_\alpha} \neq 0$ for $\alpha = 1, 2, \cdots, k$

and if

(ii) $\inf_{\mathrm{Re}\, s \geq \sigma} |\det[I + \hat{G}(s)]| > 0$,

then the closed-loop impulse response, $H(\cdot)$, is in $\mathcal{A}^{nxn}(\sigma)$.

Theorem 2. (Continuous-time) Suppose that $\hat{G}(s)$ is given by (3) and that $k = 1$ and $m_1 = 1$ (i.e. $\hat{G}$ has only a simple pole, p_1, in the closed half plane $\mathrm{Re}\ s \geq \sigma$). Suppose also that the residue matrix R_{11} is singular. Under these conditions, if[†]

(i) $\det[\hat{M}_{22}(p_1)] \neq 0$,

and if

(ii) $\inf_{\mathrm{Re}\ s \geq \sigma} |\det[I + \hat{G}(s)]| > 0$

then the closed-loop impulse response $H(\cdot)$ is in $\mathcal{A}^{nxn}(\sigma)$.

Corollary 2.1. Suppose that $\hat{G}(s)$ is given by (3) but that $k > 1$ and $m_\alpha = 1$ for $\alpha = 1, 2, \cdots, k$ (i.e. $\hat{G}(s)$ has only simple poles in $\mathrm{Re}\ s \geq \sigma$). Suppose also that

(i) either $\det R_{\alpha 1} \neq 0$

or, whenever $\det R_{\alpha 1} = 0$ we have

$$\det[\hat{M}_{22}(p_\alpha)] \neq 0,$$

and

(ii) $$\inf_{\mathrm{Re}\ s \geq \sigma} |\det[I + \hat{G}(s)]| > 0$$

Then the closed-loop impulse response H is in $\mathcal{A}^{nxn}(\sigma)$.

In the discrete-time case, the impulse response is specified as a sequence of matrices in $\mathbb{C}^{nxn}$ (or $\mathbb{R}^{nxn}$) say, $(G_0, G_1, G_2, \cdots\cdots)$. We say that a sequence belongs to $\ell^1_{nxn}(\rho)$ for some positive real number ρ iff $\sum_{k=0}^{\infty} \|G_k\| \rho^{-k} < \infty$, and we say that its corresponding z-transform $\tilde{G}(z) = \sum_0^\infty G_k z^{-k}$

is in $\tilde{\ell}^1_{nxn}(\rho)$. The analogous results of Theorem 1 for the discrete-time case can be found in [2]. We state below in Theorem 3 and Corollary 3.1 the discrete-time analogs to Theorem 2 and Corollary 2.1.

Theorem 3. (Discrete-time) Suppose that $\tilde{G}(z)$ is given by

$$\tilde{G}(z) = \sum_{0}^{\infty} G_i z^{-i} + \frac{R_{11}}{(z-p_1)}$$

$$\triangleq \tilde{G}_\ell(z) + z^{-1}(1 - p_1 z^{-1})^{-1} R_{11}$$

where

(a) $\tilde{G}_\ell(\cdot) \in \tilde{\ell}^1_{nxn}(\rho)$ for some positive real ρ,

(b) $p_1 \in \mathbb{C}$ and $|p_1| \geq \rho$

(c) $R_{11} \in \mathbb{C}^{nxn}$ is singular.

Under these conditions, if[††]

(i) $\det[\tilde{M}_{22}(p_1)] \neq 0$.

and if

(ii) $\inf_{|z| \geq \rho} |\det[I + \tilde{G}(z)]| > 0$

Then the closed-loop impulse response $H \in \ell^1_{nxn}(\rho)$.

Corollary 3.1. Suppose that $\hat{G}(z)$ is given by

$$\tilde{G}(z) = \sum_{0}^{\infty} G_i z^{-i} + \sum_{\alpha=1}^{k} \frac{R_{\alpha 1}}{(z - p_\alpha)}$$

[††] $\tilde{M}_{22}(z)$ is defined similarly as in Theorem 2.

$$\triangleq \tilde{G}_{\ell}(z) + \sum_{\alpha=1}^{k} z^{-1}(1 - p_\alpha z^{-1})^{-1} R_{\alpha 1}$$

where

(a) $\tilde{G}_{\ell}(\cdot) \in \tilde{\ell}^{1}_{nxn}(\rho)$ for some positive real ρ,

(b) for $\alpha = 1, 2, \cdots, k$, $p_\alpha \in \mathbb{C}$, $|p_\alpha| \geq \rho$, and for $\alpha \neq \alpha'$, $p_\alpha \neq p_{\alpha'}$.

Under these conditions, if

(i) either $\det R_{\alpha 1} \neq 0$

or, whenever $\det R_{\alpha 1} = 0$, we have

$$\det[\tilde{M}_{22}(p_\alpha)] \neq 0,$$

and if

(ii) $$\inf_{|z| \geq \rho} |\det[I + \tilde{G}(z)]| > 0$$

Then the closed-loop impulse response H is in $\ell^{1}_{nxn}(\rho)$.

It is expected that the final paper will include further results.

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